Questões de Álgebra Homológica

Estes são os enunciados das questões, onde tem umas dicas de como resolvê-las, encontradas no site http://laudelino.dce.ufpb.br. As dicas de como resolvê-las estão escritas em português. A numeração das questões é a que está no livro de Joseph J. Rotman, An Introductionto Homological Algebra, Springer (2008).

Questions - Homological Algebra

These are the questions where you can find some hints on how to solve them on this site http://laudelino.dce.ufpb.br. The hints are all in portuguese and handwriting. Good luck! The numbers of the questions follows the numeration of the reference book by Joseph J. Rotman, An Introductionto Homological Algebra, Springer (2008).

1.1 (i) Prove, in every category C, that each object $A \in obj(C)$ has a unique identity morphism.
(ii) If f is an isomorphism in a category, prove that its inverse is unique.

1.4 If $T : \mathcal{A} \to \mathcal{B}$ is a functor, define $T^{op} : \mathcal{A}^{op} \to \mathcal{B}$ by $T^{op}(\mathcal{A}) = T(\mathcal{A})$ for all $A \in obj(\mathcal{A})$ and $T^{op}(f^{op}) = T(f)$ for all morphisms f in $\mathcal{A}$. Prove that T^{op} is a functor having variance opposite to the variance of T.

1.6 (i) If X is a set, define FX to be the free group having basis X; that is, the elements of FX are reduced words on the alphabet X and multiplication is juxtaposition followed by cancellation. If $\varphi : X \to Y$ is a function, prove that there is a unique homomorphism $F\varphi : FX \to FY$ such that $(F\varphi)|X = \varphi$.
(ii) Prove that the $F : Sets \to Groups$ is a functor (F is called the free functor).

1.7 (i) Define C to have objects all ordered pairs (G, H), where G is a group and H is a normal subgroup of G, and to have morphisms $\varphi_* : (G, H) \to (G1, H1)$, where $\varphi : G \to G1$ is a homomorphism with $\varphi(H) \subseteq H1$. Prove that C is a category if composition in C is defined to be ordinary composition.
(ii) Construct a functor $Q : C \to Groups$ with $Q(G, H) = G/H$.
(iii) Prove that there is a functor $Groups \to Ab$ taking each group G to G/GH', where G' is its commutator subgroup.

1.11 Prove that every right R-module is a left R^{op}-module, and that every left R-module is a right R^{op}-module.

1.15 (Confused question. Needs an adaptation. Look page 23 of the reference book)

1.16 Let C be a category and let $A, B \in obj(C)$. Prove the converse of Corollary 1.18: if $A \cong B$, then $Hom_C(A, \square)$ and $Hom_C(B, \square)$ are naturally isomorphic functors.

2.4 Let $(M_i)_{i \in I}$ be a (possibly infinite) family of left R-modules and, for each i, let N_i be a submodule of M_i. Prove that

$$\left(\bigoplus_i M_i\right) / \left(\bigoplus_i N_i\right) \cong \bigoplus_i (M_i/N_i).$$

2.6 (i) Let $\to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to$ be an exact sequence, and let $\text{im} d_{n+1} = K_n = ker d_n$ for all n. Prove that

$$0 \to K_n \xrightarrow{i_n} A_n \xrightarrow{d'_n} K_{n-1} \to 0$$

is an exact sequence for all n, where i_n is the inclusion and d'_n is obtained from d_n by changing its target. We say that the original sequence has been factored into these short exact sequences.
(ii) Let

$$\to A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} K \to 0$$

and

$$0 \to K \xrightarrow{g_0} B_0 \to g_1 B_1 \to$$

be exact sequences. Prove that

$$\to A_1 \xrightarrow{f_1} A_0 \xrightarrow{g_0 f_0} B_0 \xrightarrow{g_1} B_1 \to$$

is an exact sequence. We say that the original two sequences have been spliced to form the new exact sequence.

2.7 Use left exactness of Hom to prove that if G is an abelian group, then $Hom_{\mathcal{Z}}(\mathcal{Z}_n, G) \cong G[n]$, where $G[n] = \{g \in G; ng = 0\}$.

2.9 (Too much thing to write and the answer you can find in a good book of Linear Algebra)

2.18 (It's a question from famous Atyiah's book)

2.19 Let R be a ring, let A and B be left R-modules, and let $r \in Z(R)$.
(i) If $\mu_r : B \to B$ is multiplication by r, prove that the induced map $(\mu_r)_* : Hom(A, B) \to Hom(A, B)$ is also multiplication by r.
(ii) If $m_r : A \to A$ is multiplication by r, prove that the induced map $(m_r)_* : Hom(A, B) \to HomR(A, B)$ is also multiplication by r.

2.22 (i) Prove that $Hom_{\mathbb{Z}}(\mathbb{Q}, C) = \{0\}$ for every cyclic group C.

(ii) Let R be a commutative ring. If M is an R-module such that $Hom_R(M, R/I) = \{0\}$ for every nonzero ideal I, prove that im$f \subseteq \cap I$ for every R-map $f : M \to R$, where the intersection is over all nonzero ideals I in R.

(iii) Let R be a domain and suppose that M is an R-module with $Hom_R(M, R/I) = \{0\}$ for all nonzero ideals I in R. Prove that $Hom_R(M, R) = \{0\}$.

5.1 (i) Prove that $\varnothing$ is an initial object in Sets.

(ii) Prove that any one-point set $\Omega = \{x_0\}$ is a terminal object in Sets. In particular, what is the function $\varnothing \to \Omega$?

5.2 A zero object in a category C is an object that is both an initial object and a terminal object.

(i) Prove the uniqueness to isomorphism of initial, terminal, and zero objects, if they exist.

(ii) Prove that $\{0\}$ is a zero object in RMod and that $\{1\}$ is a zero object in Groups.

(iii) Prove that neither Sets nor Top has a zero object.

(iv) Prove that if $A = \{a\}$ is a set with one element, then (A, a) is a zero object in $Sets_*$, the category of pointed sets. If A is given the discrete topology, prove that (A, a) is a zero object in Top_*, the category of pointed topological spaces.

5.3 (i) Prove that the zero ring is not an initial object in ComRings.

(ii) If k is a commutative ring, prove that k is an initial object in $ComAlg_k$, the category of all commutative k-algebras.

(iii) In ComRings, prove that $\mathbb{Z}$ is an initial object and that the zero ring $\{0\}$ is a terminal object.

5.10 (i) Given a pushout diagram in RMod (see the diagram in the book) prove that g injective implies α injective and that g surjective implies α surjective. Thus, parallel arrows have the same properties.

(ii) Given a pullback diagram in RMod (see the diagram in the book) prove that f injective implies α injective and that f surjective implies α surjective. Thus, parallel arrows have the same properties.

5.13 (This question seems to be wrong)

5.14

5.16

5.21 Let $0 \to U \to V \to V/U \to 0$ be an exact sequence of left R-modules.

(i) Let $\{Ui, \alpha^i_j\}$ be a direct system of submodules of U, where $(\alpha^i_j : U_i \to Uj)_{i \leq j}$ are inclusions. Prove that $\{V/U_i, e^i_j\}$ is a direct system, where each $e^i_j : V/U_i \to V/U_j$ is enlargement of coset.

(ii) If $\varinjlim U_i = U$, prove that $\varinjlim(V/U_i) \cong V/U$.

5.26 In RMod, let $r : \{A_i, \alpha^i_j\} \to \{B_i, \beta^i_j\}$ and $s : \{B_i, \beta^i_j\} \to \{C_i, \gamma^i_j\}$ be morphisms of inverse systems over any (not necessarily directed) index set I. If

$$0 \to A_i \xrightarrow{r_i} B_i \xrightarrow{s_i} C_i \to 0$$

is exact for each $i \in I$, prove that there are homomorphisms $\overleftarrow{r}, \overleftarrow{s}$ given by the universal property of inverse limits, and an exact sequence

$$0 \to \varprojlim A_i \xrightarrow{\overleftarrow{r}} \varprojlim B_i \xrightarrow{\overleftarrow{s}} \varprojlim C_i \to 0$$

5.31 Let (F, G) be an adjoint pair of functors between module categories. Prove that if G is exact, then F preserves projectives; that is, if P is a projective module, then FP is projective. Dually, prove that if F is exact, then G preserves injectives.

5.32 (i) Let $F : \mathit{Groups} \to \mathit{Ab}$ be the functor with $F(G) = G/G'$, where G' is the commutator subgroup of a group G, and let $U : \mathit{Ab} \to \mathit{Groups}$ be the functor taking every abelian group A into itself (that is, UA regards A as a not necessarily abelian group). Prove that (F,U) is an adjoint pair of functors.

(ii) Prove that the unit of the adjoint pair (F,U) is the natural map $G \to G/G'$.

5.55 (i) Prove that a function is epic in Sets if and only if it is surjective and that a function is monic in Sets if and only if it is injective.

(ii) Prove that an R-map is epic in RMod if and only if it is surjective and that an R-map is monic in RMod if and only if it is injective.

5.56 Let C be the category of all divisible abelian groups.

(i) Prove that the natural map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic in C.

(ii) Conclude that C is a concrete category in which monomorphisms and injections do not coincide.

5.57 Prove, in every category, that the injections of a coproduct are monic and the projections of a product are epic.

5.62 Prove that every object in Sets is projective and injective.

6.1 If C is a complex with $C_n = \{0\}$ for some n, prove that $H_n(C) = \{0\}$.

6.2 Prove that isomorphic complexes have the same homology: if C and D are isomorphic, then $H_n(C) \cong H_n(D)$ for all $n \in \mathbb{Z}$.

6.3

6.6 Let $f, g : C \to C'$ be chain maps, and let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. If $f \simeq g$, prove that $Ff \simeq Fg$; that is, if f and g are homotopic, then Ff and Fg are homotopic.

6.8 Let R and A be rings, and let $T : RMod \to AMod$ be an exact additive functor. Prove that T commutes with homology; that is, for every complex (C, d) in RComp and for every $n \in \mathbb{Z}$, there is an isomorphism $H_n(TC, Td) \cong TH_n(C, d)$.

6.12, **6.14**, **6.15**, **6.16**, **6.18**, **6.19**, **6.20**, **7.2**, **7.3**, **7.4**, **7.7**, **7.12**, **8.3**, **8.4**, **8.5**.

Exercícios Álgebra Homológica

Parte 1

1. Fronteira da fronteira de um tetraedro é zero, $\partial(\partial([v_0\ v_1\ v_2\ v_3])) = 0$.

2. Estudar a diferença entre "conjunto" e "classe". (Axiomas de Zermelo-Frankel; a definição de conjunto, cardinal, semi class, class; Exemplos)

3. Mostrar que existem correspondências 1-1 entre objetos e morfismos de R-Mod e Mod-R^{op}. Corolário: Se R for comutativo, então R-Mod=Mod-R.

4. Mostre que $1_A \in Hom_{\mathcal{C}}(A, B)$ é único.

5. Se f é um isomorfismo em uma categoria, prove que sua inversa é única.

6. Seja $T : \mathcal{C} \to \mathcal{D}$ funtor. Defina $T^{op} : \mathcal{C}^{op} \to \mathcal{D}$ por $T^{op}(A) = T(A)$, para todo $A \in obj(\mathcal{C})$ e $T^{op}(f^{op}) = T(f)$ para todo morfismo f em $\mathcal{C}$. Prove que T^{op} é um funtor que tem variância oposta a de T.

7. Se $\mathcal{C}$ é uma categoria pequena e $F, G : \mathcal{C} \to \mathcal{D}$ são funtores de uma mesma variância, prove que $Nat(F, G)$ é um conjunto (não uma classe própria).

8. Dê exemplo de categorias $\mathcal{C}, \mathcal{D}$ e funtores $S, T : \mathcal{C} \to \mathcal{D}$ tais que $Nat(S, T)$ é uma classe própria.

9. (i) Seja X um conjunto, definimos FX como sendo o grupo livre tendo base X; ou seja, os elementos de FX são palavras reduzidas no alfabeto X e multiplicação é justaposição seguida de cancelamentos. Se $\varphi : X \to Y$ é uma função, prove que existe um único homomorfismo $F\varphi : FX \to FY$ tal que $(F\varphi)|_X = \varphi$.

   (ii) Prove que F : **Sets** $\to$ **Groups** é um funtor (F é chamado de *funtor livre*).

10. (i) Defina $\mathcal{C}$ tendo como objetos todos os pares ordenados (G, H), onde G é um grupo e H um subgrupo normal de G, e tendo morfismos $\varphi_* : (G, H) \to (G_1, H_1)$, onde $\varphi : G \to H$ é homomorfismo com $\varphi(H) \subseteq H_1$. Prove que $\mathcal{C}$ é uma categoria se composição em $\mathcal{C}$ é definida como composição usual.

    (ii) Construa um funtor $Q : \mathcal{C} \to$ **Groups** com $Q(G, H) = \frac{G}{H}$.

    (iii) Prove que existe um funtor **Groups** $\to$ **Ab** levando cada grupo G em $\frac{G}{G'}$, onde G' é o subgrupo comutador de G.

11. Sejam $F, G, H : \mathcal{C} \to \mathcal{D}$ funtores de mesma variância e $\tau : F \to G$ e $\sigma : G \to H$ transformações naturais.

    (i) Prove que a composta $\sigma\tau : F \to H$ é uma transformação natural.

    (ii) Se $\tau : F \to G$ é um isomorfismo natural, defina $\sigma_C : GC \to FC$ por $\sigma_C = \tau_C^{-1}, \forall C \in \text{obj}(\mathcal{C})$. Prove que $\sigma = \tau^{-1}$ é uma transformação natural de G em F.

12. Sejam $\mathcal{C}$ uma categoria e $A, B \in \text{obj}(\mathcal{C})$. Prove a volta do Corolário 1.18 (iii): Se $A \cong B$, então $Hom_{\mathcal{C}}(A, \square)$ e $Hom_{\mathcal{C}}(B, \square)$ são naturalmente isomorfos.

Parte 2

1. Use o fato de Hom ser exato à esquerda para provar que: se G é um grupo abeliano, então $Hom_{\mathbb{Z}}(\mathbb{Z}_n, G) \cong G[n]$, onde $G[n] := \{g \in G; ng = 0\}$.

2. (i) Sejam $v_1, \ldots, v_n$ uma base de um espaço vetorial V sobre um corpo k e $v_i^* : V \to k$ definido por $v_i^* = (\Box, v)$ [ver Exemplo 1.16]. Prove que $v_1^*, \ldots, v_n^*$ é uma base V^* (chamada base dual de V).

   (ii) Sejam $f : V \to V$ uma transformação linear e A a matriz de f associada a base $v_1, \ldots, v_n$ de V. Prove que a matriz do mapa induzido $f^* : V^* \to V^*$ com a respeito a base dual é a transposta A^t de A.

3. (i) Sejam $\to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to$ uma sequência exata e $\text{Im} d_{n+1} = K_n = \ker d_n$ para todo n. Prove que $0 \to K_n \xrightarrow{i_n} A_n \xrightarrow{d'_n} K_{n-1} \to 0$ é uma sequência exata para todo n, onde i_n é a inclusão e d'_n é obtida de d_n mudando o contradomínio (alvo). Dizemos que a sequência foi fatorada nessa sequência exata curta.

   (ii) Sejam $\to A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} K \to 0$ e $0 \to K \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \to$ sequências exatas. Prove que $\to A_1 \xrightarrow{f_1} A_0 \xrightarrow{g_0 f_0} B_0 \xrightarrow{g_1} B_1 \to$ é uma sequência exata. As duas sequências originais foram "spliced" para formar a nova sequência exata.

4. Sejam $(M_i)_{i \in I}$ uma família (possivelmente infinita) de R-módulos à esquerda e, para cada i, N_i submódulo de M_i. Prove que

$$\frac{(\bigoplus_i M_i)}{(\bigoplus_i N_i)} \cong \bigoplus_i \left(\frac{M_i}{N_i} \right).$$

5. (i) Prove que $Hom_{\mathbb{Z}}(\mathbb{Q}, C) = \{0\}$ para todo grupo cíclico C.

   (ii) Seja R um anel comutativo. Se M é um R-módulo tal que $Hom_R(M, \frac{R}{I}) = \{0\}$ para todo $I \neq \{0\}$, prove que $\text{Im} f \subseteq \cap I$ para todo R-mapa $f : M \to R$, onde a interseção é sobre todos os ideais $I \neq \{0\}$.

   (iii) Seja R um domínio e suponho qua M é um R-módulo com $Hom_R(M, \frac{R}{I}) = \{0\}$ para todo ideal $I \neq \{0\}$ em R. Prove que $Hom_R(M, R) = \{0\}$. **Dica:** $r \in \cap I \Rightarrow r$ nilpotente.